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A generalized Jaynes–Cummings Hamiltonian and supersymmetric shape invariance

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Abstract. A class of shape-invariant bound-state problems which represent two-level systems are introduced. It is shown that the coupled-channel Hamiltonians obtained correspond to the generalization of the Jaynes–Cummings Hamiltonian.

1. Introduction

Supersymmetric quantum mechanics ([1], for a recent review see [2]) deals with pairs of Hamiltonians which have the same energy spectra, but different eigenstates. A number of such pairs of Hamiltonians share an integrability condition called shape invariance [3]. Although not all exactly solvable problems are shape invariant [4], shape invariance, especially in its algebraic formulation [5–7], is a powerful technique to study exactly solvable systems.

Supersymmetric quantum mechanics is generally studied in the context of one-dimensional systems. The partner Hamiltonians

$$\hat{H}_1 = \hat{A}^\dagger \hat{A} \tag{1.1a}$$

$$\hat{H}_2 = \hat{A}\hat{A}^\dagger \tag{1.1b}$$

are most readily written in terms of one-dimensional operators

$$\hat{A} \equiv W(x) + \frac{\mathrm{i}}{\sqrt{2m}}\hat{p} \tag{1.2a}$$

$$\hat{A}^\dagger \equiv W(x) - \frac{\mathrm{i}}{\sqrt{2m}}\hat{p} \tag{1.2b}$$

where $W(x)$ is the superpotential. Attempts were made to generalize supersymmetric quantum mechanics and the concept of shape invariance beyond one-dimensional and spherically symmetric three-dimensional problems. These include non-central [8], non-local [9] and periodic [10] potentials; a three-body problem in one dimension [11] with a three-body force [12]; N-body problem [13]; and coupled-channel problems [14, 15]. It is not easy

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to find exact solutions to these problems. For example, in the coupled-channel case a general shape invariance is only possible in the limit where the superpotential is separable [15] which corresponds to the well known sudden approximation in the coupled-channel problem [16]. Our goal in this paper is to introduce a class of shape-invariant coupled-channel problems which correspond to the generalization of the Jaynes–Cummings Hamiltonian [17].

2. Shape invariance

The Hamiltonian $\hat{H}_1$ of equation (1.1) is called shape invariant if the condition

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1) \tag{2.1}$$

is satisfied [3]. In this equation a_1 and a_2 represent parameters of the Hamiltonian. The parameter a_2 is a function of a_1 and the remainder $R(a_1)$ is independent of the dynamical variables such as position and momentum. As written the condition of equation (2.1) does not require the Hamiltonian to be one dimensional, and one does not need to choose the ansatz of equation (1.2). In the cases studied so far the parameters a_1 and a_2 are either related by a translation [4, 18] or a scaling [19]. Introducing the similarity transformation that replaces a_1 with a_2 in a given operator

$$T(\hat{a}_1)\hat{O}(a_1)\hat{T}^\dagger(a_1) = \hat{O}(a_2) \tag{2.2}$$

and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1) \tag{2.3}$$

$$\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1) \tag{2.4}$$

the Hamiltonians of equation (1.1) take the forms

$$\hat{H}_1 = \hat{B}_+\hat{B}_- \tag{2.5}$$

and

$$\hat{H}_2 = \hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger. \tag{2.6}$$

Using equation (2.1) one can also easily prove the commutation relation [5]

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1)R(a_1)\hat{T}(a_1) \equiv R(a_0) \tag{2.7}$$

where we have used the identity

$$R(a_n) = \hat{T}(a_1)R(a_{n-1})\hat{T}^\dagger(a_1) \tag{2.8}$$

valid for any n. The ground state of the Hamiltonian $\hat{H}_1$ satisfies the condition

$$\hat{A}|\psi_0\rangle = 0 = \hat{B}_-|\psi_0\rangle. \tag{2.9}$$

The nth excited state of $\hat{H}_1$ is given by

$$|\psi_n\rangle \sim (\hat{B}_+)^n|\psi_0\rangle \tag{2.10}$$

with the eigenvalue

$$\varepsilon_n = \sum_{k=1}^{n} R(a_k). \tag{2.11}$$

Note that the eigenstate of equation (2.10) needs to be suitably normalized. We discuss the normalization of this state in the next section.

3. Generalization of the Jaynes–Cummings Hamiltonian

To generalize the Jaynes–Cummings Hamiltonian to general shape-invariant systems we introduce the operator

$$\hat{S} = \sigma_+ \hat{A} + \sigma_- \hat{A}^\dagger \tag{3.1}$$

where

$$\sigma_\pm = \tfrac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2) \tag{3.2}$$

with σ_i, $i = 1$, 2 and 3, being the Pauli matrices and the operators $\hat{A}$ and $\hat{A}^\dagger$ satisfy the shape-invariance condition of equation (2.1). We search for the eigenstates of $\hat{S}$. It is more convenient to work with the square of this operator, which can be written as

$$\hat{S}^2 = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \hat{B}_-\hat{B}_+ & 0 \\ 0 & \hat{B}_+\hat{B}_- \end{bmatrix} \begin{bmatrix} \hat{T}^\dagger & 0 \\ 0 & \pm 1 \end{bmatrix}. \tag{3.3}$$

Note the freedom of choice of the sign in this equation, which results in two possible decompositions of $\hat{S}^2$.

We next introduce the states

$$|\Psi\rangle_\pm = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} |m\rangle \\ |n\rangle \end{bmatrix} \tag{3.4}$$

where $|m\rangle$ and $|n\rangle$ are the abbreviated notation for the states $|\psi_n\rangle$ and $|\psi_m\rangle$ of equation (2.10). Using equations (2.7), (3.3) and (3.4) and the fact that the operator $\hat{T}$ is unitary one obtains

$$\begin{aligned} \hat{S}^2|\Psi\rangle_\pm &= \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \hat{B}_+\hat{B}_- + R(a_0) & 0 \\ 0 & \hat{B}_+\hat{B}_- \end{bmatrix} \begin{bmatrix} |m\rangle \\ |n\rangle \end{bmatrix} \\ &= \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \varepsilon_m + R(a_0) & 0 \\ 0 & \varepsilon_n \end{bmatrix} \begin{bmatrix} |m\rangle \\ |n\rangle \end{bmatrix}. \end{aligned} \tag{3.5}$$

Using equations (2.8) and (2.11) one can write

$$\begin{aligned} \hat{T}\,[\varepsilon_m + R(a_0)]\,\hat{T}^\dagger &= \hat{T}\,[R(a_1) + R(a_2) + \cdots + R(a_m) + R(a_0)]\,\hat{T}^\dagger \\ &= R(a_2) + R(a_3) + \cdots + R(a_{m+1}) + R(a_1) = \varepsilon_{m+1}. \end{aligned} \tag{3.6}$$

Hence the states

$$|\Psi_m\rangle_\pm = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} |m\rangle \\ |m+1\rangle \end{bmatrix} \qquad m = 0, 1, 2, \ldots \tag{3.7}$$

are the normalized eigenstates of the operator $\hat{S}^2$

$$\hat{S}^2|\Psi_m\rangle_\pm = \varepsilon_{m+1}|\Psi_m\rangle_\pm. \tag{3.8}$$

One can also calculate the action of the operator $\hat{S}$ on this state

$$\hat{S}|\Psi_m\rangle_\pm = \frac{1}{\sqrt{2}} \begin{bmatrix} \pm\hat{T}\hat{B}_-|m+1\rangle \\ \hat{B}_+|m\rangle \end{bmatrix}. \tag{3.9}$$

Introducing the operator [7]

$$\hat{Q}^\dagger = (\hat{B}_+\hat{B}_-)^{-1/2}\hat{B}_+ \tag{3.10}$$

one can write the normalized eigenstate of $\hat{H}_1$ as

$$|m\rangle = (\hat{Q}^\dagger)^m|0\rangle. \tag{3.11}$$

Using equations (3.10) and (3.11) one obtains

$$\hat{B}_+|m\rangle = \sqrt{\varepsilon_{m+1}}|m+1\rangle. \tag{3.12}$$

Similarly,

$$\begin{aligned}\hat{T}\hat{B}_-|m+1\rangle &= \hat{T}\hat{B}_-\frac{1}{\sqrt{\hat{B}_+\hat{B}_-}}\hat{B}_+|m\rangle \\ &= \hat{T}\sqrt{\hat{B}_-\hat{B}_+}|m\rangle \\ &= \hat{T}\sqrt{\varepsilon_m + R(a_0)}|m\rangle \\ &= \sqrt{\varepsilon_{m+1}}\hat{T}|m\rangle.\end{aligned} \tag{3.13}$$

Using equations (3.12) and (3.13), equation (3.9) takes the form

$$\begin{aligned}\hat{S}|\Psi_m\rangle_\pm &= \frac{1}{\sqrt{2}}\sqrt{\varepsilon_{m+1}}\begin{bmatrix}\pm\hat{T}|m\rangle \\ |m+1\rangle\end{bmatrix} \\ &= \pm\sqrt{\varepsilon_{m+1}}|\Psi_m\rangle_\pm.\end{aligned} \tag{3.14}$$

Equations (3.8) and (3.14) indicate that the Hamiltonian

$$\hat{H} = \hat{S}^2 + \sqrt{\hbar\Omega}\hat{S} \tag{3.15}$$

where Ω is a constant, has the eigenstates $|\Psi_m\rangle_\pm$

$$\hat{H}|\Psi_m\rangle_\pm = (\varepsilon_{m+1} \pm \sqrt{\hbar\Omega}\sqrt{\varepsilon_{m+1}})|\Psi_m\rangle_\pm \tag{3.16}$$

with the exception of the ground state. It is easy to show that the ground state is

$$|\Psi_0\rangle = \begin{bmatrix}0 \\ |0\rangle\end{bmatrix} \tag{3.17}$$

with eigenvalue 0. To emphasize the structure of equation (3.16) as the generalized Jaynes–Cummings Hamiltonian we rewrite it as

$$\hat{H} = \hat{A}^\dagger\hat{A} + \tfrac{1}{2}[\hat{A}, \hat{A}^\dagger](\sigma_3 + 1) + \sqrt{\hbar\Omega}(\sigma_+\hat{A} + \sigma_-\hat{A}^\dagger). \tag{3.18}$$

This Hamiltonian represents a number of systems. When $\hat{A}$ describes the annihilation operator for the harmonic oscillator, $[\hat{A}, \hat{A}^\dagger] = \hbar\omega$, where ω is the oscillator frequency. In this case equation (3.18) reduces to the standard Jaynes–Cummings Hamiltonian. When $\hat{A}^\dagger\hat{A}$ describes the Morse–Hamiltonian, equation (3.18) takes the form

$$\begin{aligned}\hat{H} = \frac{\hat{p}^2}{2M} &+ V_0\left(e^{-2\lambda x} - 2e^{-\lambda x}\right) + \sqrt{V_0}\frac{\hbar\lambda}{\sqrt{2M}}(\sigma_3 + 1)\,e^{-\lambda x} \\ &+ \sqrt{\hbar\Omega V_0}\left[\sigma_1\left(1 - \frac{\hbar\lambda}{2\sqrt{2MV_0}} - e^{-\lambda x}\right) - \sigma_2\frac{\hat{p}}{\sqrt{2MV_0}}\right]\end{aligned} \tag{3.19}$$

with the energy eigenvalues

$$\begin{aligned}E_m = \sqrt{V_0}&\frac{\hbar\lambda}{\sqrt{2M}}(m+1)\left[2 - \frac{\hbar\lambda}{\sqrt{2MV_0}}(m+2)\right] \\ &\pm\left\{\hbar\Omega\sqrt{V_0}\frac{\hbar\lambda}{\sqrt{2M}}(m+1)\left[2 - \frac{\hbar\lambda}{\sqrt{2MV_0}}(m+2)\right]\right\}^{1/2}.\end{aligned} \tag{3.20}$$

Both the harmonic oscillator and Morse potential are shape-invariant potentials where parameters are related by a translation. It is also straightforward to use those shape-invariant potentials where the parameters are related by a scaling [19] in writing down equation (3.18).

4. Conclusions

In this paper we have introduced a class of shape-invariant bound-state problems which represent two-level systems. The corresponding coupled-channel Hamiltonians generalize the Jaynes–Cummings Hamiltonian. If we take $\hat{H}_1$ to be the simplest shape-invariant system, namely the harmonic oscillator, our Hamiltonian, equation (3.18), reduces to the standard Jaynes–Cummings Hamiltonian, which has been used extensively to model a single field mode on resonance with atomic transitions. For a general shape-invariant system equation (3.18) represents a non-trivial coupled-channels problem which may find applications in molecular, atomic or nuclear physics.

In this paper we only addressed generalization of the Jaynes–Cummings model to other shape-invariant bound-state systems. Supersymmetric quantum mechanics has been applied to alpha particle [20] and Coulomb [21] scattering problems. More recently shape invariance was utilized to calculate quantum tunnelling probabilities [22]. It may be possible to generalize our results to such continuum problems. Such an investigation will be deferred to a later publication.

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